MIP Models for the Transmitter Clustering Problem

Paul A. Rubin

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Problem source: "How to perform clustering of two different sets of entities?" (Operations Research Stack Exchange)

Notation common to both models

- $t \in \{1, \ldots, T\}$ indexes a transmitter;
- $u \in \{1, \ldots, U\}$ indexes a user;
- $w_{t,u} \geq 0$ is the weight between transmitter t and user u;
- $C \in \mathbb{Z}^+$ is the cluster size limit (maximum number of transmitters allowed in a cluster);
- $L \in \mathbb{Z}^+$ is the maximum number of clusters allowed;
- $q_u \geq 0$ is the service quality for user u;and
- $\tau(u) \in \{1, \ldots, T\}$ is the transmitter with highest weight for user u.

If there is no explicit limit to the number of clusters, set $L = T$.
The formula for service quality is

$$q_u = \frac{\sum_{t \in K(u)} w_{t,u}}{\sum_{t \notin K(u)} w_{t,u}} \tag{1}$$

where $K(u)$ is the cluster containing user u. We will need an upper bound $\overline{Q}_u$ for q_u. which can be computed by evaluating (1) when $K(u)$ consists of the C transmitters with highest weights for u.

Model 1

The first model is based on binary variables for each pair of transmitters, indicating whether they belong to the same cluster. Each populated cluster will contain one transmitter designated as its “anchor”. The model contains the following variables:

- $x_{t,t'} \in \{0, 1\}$ for all $t, t' \in \{1, \ldots, T\}$, equal to 1 when transmitters t and t' belong to the same cluster if $t \neq t'$ and indicating whether t anchors its cluster when $t = t'$;
- $y_{t,u} \in \{0, 1\}$ for all $t \in \{1, \ldots, T\}$ and $u \in \{1, \ldots, U\}$, equal to 1 if transmitter t and user u belong to the same cluster;
- $q_u \in \left[0, \overline{Q}_u\right]$ for all $u \in \{1, \ldots, U\}$, representing the service quality achieved by user u;
- $v_{t,t'} \in [0, 1]$ for all $t, t' \in \{1, \ldots, T\}$ with $t \neq t'$, used to force each cluster to have an anchor (and explained below); and
- $z_{t,u} \in \left[0, \overline{Q}_u\right]$ for all $t \in \{1, \ldots, T\}$ and $u \in \{1, \ldots, U\}$,used to linear (1).

The objective function is

$$\max \sum_{u=1}^{U} q_u. \tag{2}$$

The constraints are as follows.

- Membership of transmitters in the same cluster is symmetric.

$$x_{t,t'} = x_{t',t} \ \forall t \neq t' \tag{3}$$

- Membership in the same cluster is transitive. If t and t' are in the same cluster and t' and t'' are in the same cluster, then t and t'' are in the same cluster.

$$x_{t,t''} \geq x_{t,t'} + x_{t',t''} - 1 \ \forall t, t', t'' \ni t \neq t' \& t' \neq t'' \& t \neq t'' \tag{4}$$

- Every transmitter must belong to a cluster containing an anchor. (We will use the anchors to count the number of clusters.)

$$v_{t,t'} \leq x_{t,t'} \ \forall t \neq t' \tag{5}$$
$$v_{t,t'} \leq x_{t',t'} \ \forall t \neq t' \tag{6}$$
$$x_{t,t} + \sum_{t' \neq t} v_{t,t'} = 1 \tag{7}$$

  So if t is an anchor $(x_{t,t} = 1)$, $v_{t,t'} = 0$ for all $t \neq t'$; but if t is not an anchor $(x_{t,t} = 0)$, then there must be exactly one $t' \neq t$ for which $v_{t,t'} = 1$, and that forces $x_{t,t'} = 1$ (t and t' are in the same cluster) and $x_{t',t'} = 1$ (t' is an anchor).

- The number of clusters is limited, which we accomplish by limiting the number of anchors.

$$\sum_{t} x_{t,t} \leq L. \tag{8}$$

- The size of each cluster is limited.

$$\sum_{t'=1}^{T} x_{t,t'} - x_{t,t} \leq C - 1 \ \forall t \tag{9}$$

  This says that the number of transmitters other than t in its cluster cannot exceed $C - 1$.

- Each cluster should have at most one anchor.

$$x_{t,t} + x_{t',t'} + x_{t,t'} \leq 2 \ \forall t < t' \tag{10}$$

  This says that if t and t' are in the same cluster $(x_{t,t'} = 1)$ then at most one of them can be an anchor.

- Every user must be in the same cluster as its "favorite" transmitter.

$$y_{\tau(u),u} = 1 \ \forall u \tag{11}$$

- Other transmitters are clustered with user u if and only if they are clustered with the "favorite" transmitter of u.

$$y_{t,u} = x_{t,\tau(u)} \ \forall u, \forall t \neq \tau(u) \tag{12}$$

- Up to this point, solutions to the model suffer from some symmetry: if you take a solution and change which transmitter in a particular cluster is considered the anchor, you get a nominally different but functionally identical solution (with the same objective value). We avoid that by forcing the transmitter in the cluster with lowest index to be the anchor.

$$x_{t',t'} \leq 1 - x_{t,t'} \; \forall t < t' \tag{13}$$

  This says that if t' is in a cluster with a transmitter having smaller index t, then t' cannot be the anchor.

- We linearize the definition of q_u in (1) by multiplying both sides by the denominator, obtaining $\sum_{t \notin K(u)} w_{t,u} q_u = \sum_{t \in K(u)} w_{t,u}$. Using $y_{t,u}$ as the indicator for whether t is in $K(u)$, this becomes $\sum_{t=1}^{T} w_{t,u} \left(1 - y_{t,u}\right) q_u = \sum_{t=1}^{T} w_{t,u} y_{t,u}$. We linearize the product on the left using the auxiliary variables $z_{t,u}$ as follows.

$$z_{t,u} \leq \overline{Q}_u y_{t,u} \; \forall t, u \tag{14}$$

$$z_{t,u} \leq q_u + \overline{Q}_u \left(1 - y_{t,u}\right) \; \forall t, u \tag{15}$$

$$z_{t,u} \geq q_u - \overline{Q}_u \left(1 - y_{t,u}\right) \; \forall t, u \tag{16}$$

$$\sum_{t=1}^{T} w_{t,u} \left(q_u - z_{t,u}\right) = \sum_{t=1}^{T} w_{t,u} y_{t,u} \; \forall u \tag{17}$$

  The first three constraints force

$$z_{t,u} = \begin{cases} q_u & y_{t,u} = 1 \\ 0 & y_{t,u} = 0 \end{cases}$$

  and the last is our linearized version of (1).

Model 2

Rather than using binary variables to indicate whether pairs of transmitters belong to the same cluster, the second model relies on binary variables indicating whether each transmitter belongs to each of L clusters. Note that some of the L clusters will likely end up empty. Variables $y_{t,u}$, $z_{t,u}$ and q_u are unchanged. The modified variables are as follows:

- $x_{t,c} \in \{0,1\}$ for all $t \in \{1, \ldots, T\}$ and $c \in \{1, \ldots, L\}$, equal to 1 if transmitter t belongs to cluster c; and
- $v_{t,u,c} \in [0,1]$ for all $t \in \{1, \ldots, T\}$, $u \in \{1, \ldots, U\}$ and $c \in \{1, \ldots, L\}$, equal to 1 if transmitter t and user u both belong to cluster c.

The objective function remains (2). Constraints (11) and (14)-(17) are retained, and the following new constraints are added.

- Every transmitter is assigned to exactly one cluster.

$$\sum_{c=1}^{L} x_{t,c} = 1 \; \forall t \tag{18}$$

- Cluster capacity is limited.

$$\sum_{t=1}^{T} x_{t,c} \leq C \; \forall c \tag{19}$$

- To mitigate symmetry resulting from numbering clusters without fundamentally changing the solution, we require that the first transmitter be assigned to the first cluster. We also allow a transmitter $t > 1$ to be assigned to a cluster $c > 1$ only if the preceding cluster $c - 1$ contains at least one lower-indexed transmitter.

$$x_{1,1} = 1 \tag{20}$$

$$x_{t,c} \le \sum_{i=1}^{t-1} x_{i,c-1} \ \forall t > 1, \forall c > 1 \tag{21}$$

- We define $v_{t,u,c}$ to be 1 if an only if both transmitter t and transmitter $\tau(u)$ belong to cluster c.

$$v_{t,u,c} \le x_{\tau(u),c} \ \forall t, u, c \tag{22}$$

$$v_{t,u,c} \le x_{t,c} \ \forall t, u, c \tag{23}$$

$$v_{t,u,c} \ge x_{\tau(u),c} + x_{t,c} - 1 \ \forall t, u, c \tag{24}$$

- Transmitter t serves user u if and only if they are in the same cluster.

$$y_{t,u} = \sum_{c=1}^{L} v_{t,u,c} \ \forall t, u \tag{25}$$

In limited computational experiments, model 2 proved to be much larger than model 1 in all dimensions (number of rows, number of columns, number of binary variables, number of nonzero matrix entries), and was slower in reducing the optimality gap, even with rather tight value for the number of clusters (L).