Implicit Hitting Set Problems

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April 15, 2016

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Thank you

INFORMS Student Chapter

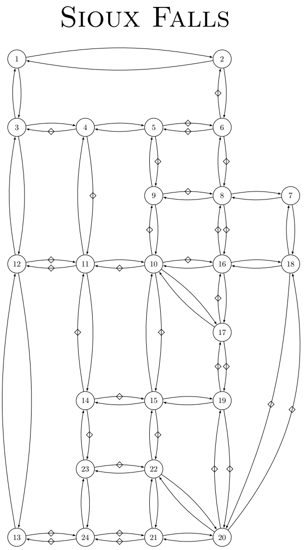
at the

University of Louisville

for the invitation to speak today.

Some Seemingly Unrelated Problems

Placing Toll Booths (Bai and Rubin, 2009)



- Mission: use tolls to induce drivers to follow "system optimal" routes
  - Road system modeled as a flow network
  - Toll booths placed on arcs
- Key decisions: where to place toll booths (choose a subset of arcs)
- Criterion: minimize cost of toll booths

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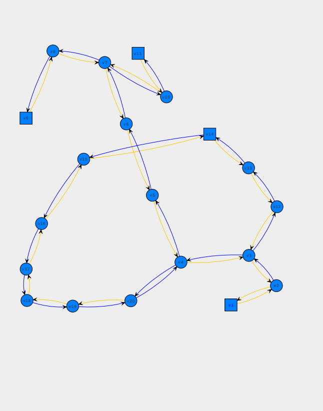
Placing Traffic Sensors (Morrison and Rubin, 2015)



- Mission: count traffic
  - Road system modeled as a flow network
  - Traffic counting sensors placed at nodes
  - **Turning ratios** known at count time
- Key decisions: where to place sensors (choose a subset of nodes)
- Criterion: minimize cost of sensors

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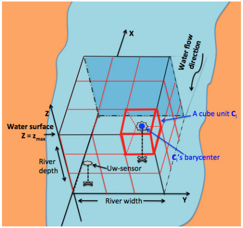
Placing Pollution Sensors (Khalfallah et al., 2016)



- Mission: detect pollution events in a body of water
  - River/lake modeled as a tessellated cube
  - Locations (sub-cubes) can be viewed as a graph
- Key decisions: where to place sensors (and one uplink) (choose a subset of nodes)
- Criterion: minimize cost of sensors

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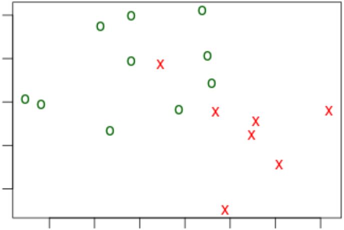
Two Group Classification [Rubin, unpublished]



- Related to Rubin (1997)
- Mission: classify observations into one of two groups
  - Linear or low degree polynomial
  - Fraudulent v. legitimate transaction
  - Cancerous v. non-cancerous tumor
- Key decisions: which observations (if any) to misclassify (choose a subset of each sample)
- Criterion: minimize misclassification costs
  - Some errors may be more expensive than others
  - Cancer screen: false positive v. false negative

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Minimum Weight IIS Cover (Parker and Ryan, 1996; Chinneck, 1996)

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

$$\Longrightarrow$$

$$x \in \emptyset$$

- IIS = irreducible infeasible subsystem
- Mission: find a set of constraints whose relaxation makes an infeasible linear program feasible
- Key decisions: which constraints to relax (choose a subset of constraints)
- Criterion: minimize the "cost" of the relaxed constraints
    - Parker and Ryan (1996) weight the constraints
    - Chinneck (1996) looks for minimum cardinality

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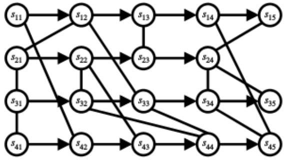
Maximum Weight Trace Problem (Moreno-Centeno and Karp, 2013)



(source: Moreno-Centeno and Karp (2013))

- Mission: find a multigenome alignment most consistent with given pairings of nucleotide subsequences
- Key decisions: which edges from an "alignment graph" to select (choose a subset of edges)
- Criterion: minimize the weight of the selected edges
    - *Read the paper; don't ask me!*

Common Elements

- Primary decisions select a cheapest set of "objects" (locations, observations, . . .)
    - 0-1 variables
- Selection must meet some adequacy conditions
- May need secondary decisions, related constraints to determine adequacy

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Secondary Variables

toll booths	• tolls to charge (continuous variables) • Lagrange multipliers (*don't ask*)
traffic sensors	• none
pollution sensors	• packet flows from sensors to other sensors/uplink (gives communication paths)
classification	• coefficients of a classifier function (linear, polynomial, . . .)
IIS cover	• original LP variables (or dual variables)
genomes	• none

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Adequacy Conditions

toll booths	• tolls, multipliers must satisfy system of linear inequalities
traffic sensors	• solution to network flow equations must be unique
pollution sensors	• sufficient detection probability at each node • sensors path-connected to uplink
classification	• correctly classify non-selected observations
IIS covering	• reduced constraint system must be feasible
genomes	• eliminate all “mixed cycles”

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Hitting Set Problems

Set Covering Problem (SCP)

- Given: objects, sets of objects
  - Sets have associated costs
- Find: cheapest collection of sets covering all objects
  - Every object belongs to at least one selected set

Sets

$$
\text{Objects} \quad
\begin{array}{r}
\\
i \rightarrow \\
\\
\\
\\
\end{array}
\left|
\begin{array}{cccccc}
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \textcircled{1} & \cdots & 1 \\
0 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 1 & 0 & \cdots & 0
\end{array}
\right|
\begin{array}{l}
\geq 1 \\
\geq 1 \\
\geq 1 \\
\vdots \\
\geq 1
\end{array}
$$

$\uparrow$
j

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Explicit Hitting Set Problem (EHSP)

- Given: objects, sets of objects (“circuits”)
  - ~~Sets~~ Objects have associated costs
- Find: cheapest collection of ~~sets~~ objects ~~covering~~ hitting all ~~objects~~ circuits
  - Every ~~object~~ circuit ~~belongs to~~ contains at least one selected ~~set~~ object

~~Sets~~ Objects

~~Objects~~ Circuits

$$
\begin{array}{r|cccccc|l}
 & 0 & 1 & 1 & 1 & \cdots & 1 & \geq 1 \\
i \rightarrow & 1 & 0 & 0 & \textcolor{red}{1} & \cdots & 1 & \geq 1 \\
 & 0 & 1 & 1 & 1 & \cdots & 1 & \geq 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 1 & 1 & 0 & \cdots & 0 & \geq 1 \\
\end{array}
$$

$\uparrow$

j

If It Walks Like A Duck, Quacks Like A Duck . . .

SCP, EHSP are mathematically equivalent.

> *"The EHSP is identical to the classic weighted set-cover problem, except that the roles of sets and elements are interchanged." (Moreno-Centeno and Karp, 2013)*

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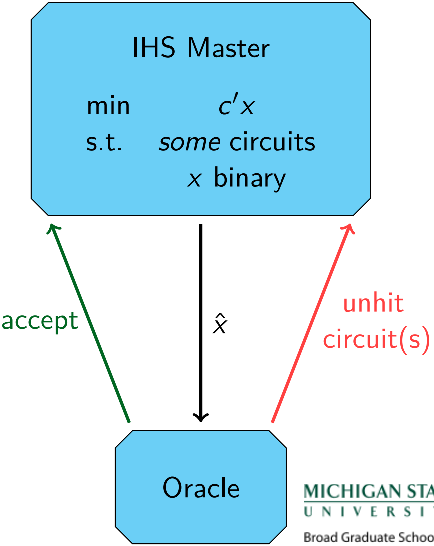
Implicit Hitting Set Problem (IHSP)

- Defined in Moreno-Centeno and Karp (2013)
- Objects (and costs) are specified at the outset
- Circuits are *not* known at the outset
    - Specified by a polynomial-time **separation oracle**
    - Oracle examines a hitting set, either blesses it or returns an unhit circuit



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IHSP Generalizations

- Do not require oracles to be polynomial time
- Optionally generate more than one circuit per candidate hitting set
- Use more than one oracle
  - Series or parallel?
  - Run all oracles to completion or stop as soon as one finds an unhit circuit?
  - At least one oracle must be definitive
    - Others can fail to find an unhit circuit when one exists
- Include some constraints at outset (not necessarily circuits)

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(Slight) Tangent: Benders Decomposition

Benders Decomposition (Benders, 1962)

- Start with an optimization model containing an embedded LP

$$\min_{x\in\mathbb{X},\, y\in\Re^n} \left\{ f(x) + c'y \mid F(x) + Ay \geq b \right\}$$

  - continuous variables (y) appear linearly
  - $f()$, $F()$ might be linear or nonlinear
  - $\mathbb{X}$ might or might not be discrete

- Isolate the LP elements in a subproblem (dependent on the value of x)

$$\min_{y\in\Re^n} \left\{ c'y \mid Ay \geq b - F(x) \right\}$$

- Master problem contains a surrogate variable (z) for $c'y$

$$\min_{x\in\mathbb{X}} \left\{ f(x) + z \,\middle|\, H \begin{bmatrix} x \\ z \end{bmatrix} \geq h \right\}$$

  - H and h come from cuts generated by the subproblem
  - Our interest is MILP ($F()$, $f()$ linear, $\mathbb{X}$ discrete)

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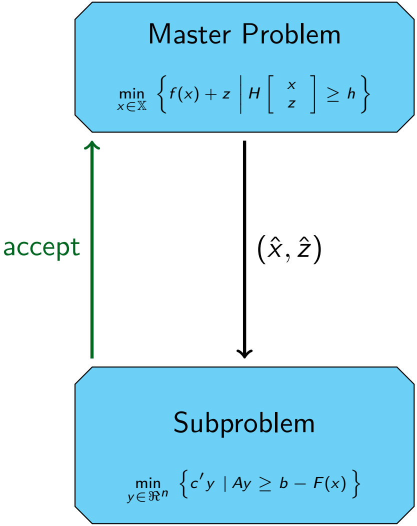
Benders Cuts (I)



- Pass each candidate master incumbent $(\hat{x}, \hat{z})$ to the subproblem
- If y exists such that $Ay \geq b - F(\hat{x})$ and $\hat{z} \geq c'y$, accept $(\hat{x}, \hat{z})$

Benders Cuts (II)

- If LP is feasible but $\hat{z} < c'y^*$ for optimal y^*, add a new **optimality (point) cut**

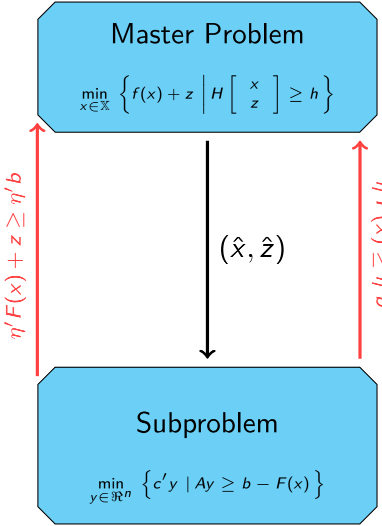
$$\eta' F(x) + z \geq \eta' b$$

to the master, where η is an optimal dual solution to the LP with $x = \hat{x}$

- If LP is infeasible, add a new **feasibility (ray) cut**

$$\eta' F(x) \geq \eta' b$$

where η is an unbounded dual ray when $x = \hat{x}$



Extensions to Benders

- More general subproblems (not necessarily LPs)
  - May not use dual rays in feasibility cuts
  - Can fall back on "no good" constraint

$$\sum_{i:\hat{x}_i=0} x_i + \sum_{i:\hat{x}_i=1} (1 - x_i) \geq 1$$

  or possibly

$$\sum_{i:\hat{x}_i=0} x_i \geq 1$$

- May use callbacks to generate Benders cuts from each incumbent
  - "Traditional" method: solve master to optimality, generate cut, rinse and repeat
- Multiple subproblems
  - Stop with first cut or cycle through them?

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Another (Slight) Tangent: On/Off Constraints

Big M

- Some models use binary variables to turn constraints "on" and "off"
- Leads to dreaded "big M" constraints such as

$$a'y + Mx \geq b$$

  - x is binary, M is a "big constant"
  - active when $x = 0$ ($a'y + Mx \geq b \rightarrow a'y \geq b$)
  - relaxed when $x = 1$ ($a'y + Mx \geq b \rightarrow a'y + \infty \geq b$)
- M too small: model is incorrect
  - optimal solution may look infeasible
- M too big: numerical instability
  - multiple pathologies

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Combinatorial Benders Cuts (Codato and Fischetti, 2006)

- Replace M with Benders decomposition
- Master problem contains binary variables controlling constraints
- In subproblem, for given $\hat{x}$

$$a'y + Mx \geq b \longrightarrow \begin{cases} a'y \geq b & \text{if } \hat{x} = 0 \\ \text{omitted} & \text{if } \hat{x} = 1 \end{cases}$$

- No numerical issues

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Connection to the IHSP

- If $c = 0$ (y not in original objective function), Benders decomposition produces only feasibility cuts
- If feasibility cuts look like circuits (sum of binaries ≥ 1), Benders decomposition resembles an implicit hitting set approach
- Takeaway: applications of combinatorial Benders may tend to resemble IHSP

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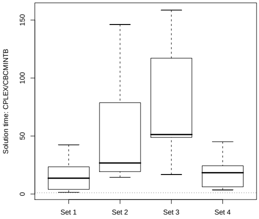
Applying the Implicit Hitting Set Approach

Toll Booths



Network	Lower bound	CPU time (secs.)
Sioux Falls	32 (0.0%)	103.6
Hull	39 (0.0%)	3,317.9
Stockholm	102.3 (31.8%)	*100,000*

- Implementation
  - Master problem is a pure IHP (or SCP)
    - No additional constraints
  - Multiple sequential oracles
    - Negative cycle detection (fast, not definitive)
    - Linear program (not definitive)
    - Sequence of LPs (definitive)
- Results
  - Smoked full (big-M) MIP model ...
  - ... but still suffered Stockholm syndrome

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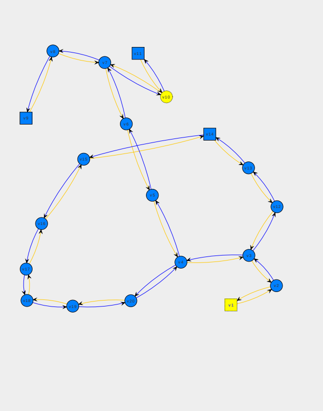
Traffic Sensors



- Implementation
  - Master problem is a pure IHP
  - Heuristic to seed master with initial circuit(s)
  - Oracles are MIP models (so far)
    - One uses Benders decomposition
- Results
  - Works on small examples
  - Need to improve scaling
    - Oracles can be very fast or very slow
    - Circuits can be high cardinality

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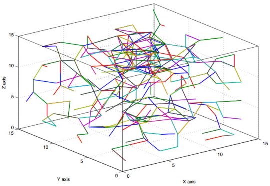
Pollution Sensors



- Implementation
  - Master problem is IHP plus detection sensitivity constraints
  - Oracles are either flow LPs or shortest path problems (ensure network connectivity)
- Results
  - Seems slower than single MIP (with *tight* estimate of M)
  - Not fully tested

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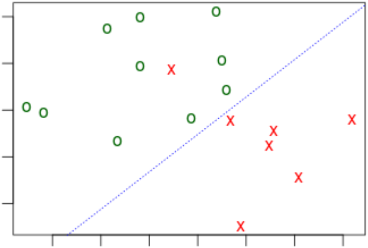
Classification



- Implementation
  - Master problem is pure IHP
  - Oracle is LP
    - Accept if LP is feasible, else
    - Circuit = no-good constraint, or
    - Circuit = IIS
- Results
  - Untested
  - Hard for any MIP to beat support vector machines (SVMs)

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IIS Cover

$$\begin{array}{rl} \text{minimize} & c'x \\ \text{subject to} & Ax \leq b \end{array}$$

$$\downarrow$$

$$\begin{array}{rl} \text{minimize} & c'x \\ \text{subject to} & \hat{A}x \leq \hat{b} \end{array}$$

- Implementation
  - Not technically an IHSP (predates Moreno-Centeno and Karp) but equivalent
  - Master problem seeded with some circuits (IISs)
  - LP oracle with varying objective functions
- Results
  - Works
  - No benchmarks

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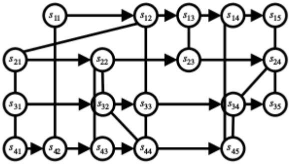
Genomes



(source: Moreno-Centeno and Karp (2013))

- Implementation
  - Master problem is pure IHP
  - Two oracles
    - Both network search algorithms
    - First faster, second definitive
    - Both randomized
    - Used in series
- Results
  - No benchmarks
  - Asserted to be fast

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Genomes

Quoting Moreno-Centeno and Karp (2013):

> *"The algorithm performed extremely well on the real-data problems (optimally solving most of the instances in less than one minute). Indeed, for practical multisequence alignment problems, our algorithm was able to optimally solve the vast majority of the problems within one hour."*

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Conclusion

For problems requiring selection of a subset of objects, either the implicit hitting set approach or (combinatorial?) Benders decomposition may be useful . . .

. . . but there are no guarantees.

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Questions $\longrightarrow$ me

Complaints$\longrightarrow$ Professor Bai

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